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Nonoscillation and oscillation of second order half-linear differential equations

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Abstract

We study the oscillation problems for the second order half-linear differential equation $[p(t)\Phi(x')]' + q(t)\Phi(x) = 0$, where $\Phi(u) = |u|^{r-1}u$ with $r > 0$, $1/p$ and q are locally integrable on $\mathbb{R}_+$; $p > 0$, $q \geqslant 0$ a.e. on $\mathbb{R}_+$, and $\int_0^\infty p^{-1/r}(t)\,dt = \infty$. We establish new criteria for this equation to be nonoscillatory and oscillatory, respectively. When $p \equiv 1$, our results are complete extensions of work by Huang [C. Huang, Oscillation and nonoscillation for second order linear differential equations, J. Math. Anal. Appl. 210 (1997) 712–723] and by Wong [J.S.W. Wong, Remarks on a paper of C. Huang, J. Math. Anal. Appl. 291 (2004) 180–188] on linear equations to the half-linear case for all $r > 0$. These results provide corrections to the wrongly established results in [J. Jiang, Oscillation and nonoscillation for second order quasilinear differential equations, Math. Sci. Res. Hot-Line 4 (6) (2000) 39–47] on nonoscillation when $0 < r < 1$ and on oscillation when $r > 1$. The approach in this paper can also be used to fully extend Elbert's criteria on linear equations to half-linear equations which will cover and improve a partial extension by Yang [X. Yang, Oscillation/nonoscillation criteria for quasilinear differential equations, J. Math. Anal. Appl. 298 (2004) 363–373].

Keywords: Second order; Half-linear differential equations; Oscillation; Nonoscillation

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1. Introduction

Let $\Phi(u) = |u|^{r-1}u$ for $r > 0$. Consider the second order half-linear differential equation

$$\left[p(t)\Phi(x')\right]' + q(t)\Phi(x) = 0, \tag{1.1}$$

where $1/p, q \in L_{\text{loc}}(\mathbb{R}_+)$, the set of locally integrable functions on $\mathbb{R}_+ := [0, \infty)$; $p > 0$, $q \geqslant 0$ a.e. on $\mathbb{R}_+$, and $\int_0^\infty p^{-1/r}(t)\,dt = \infty$. We observe that the function Φ satisfies the following properties:

(a) $\Phi(uv) = \Phi(u)\Phi(v)$,
(b) $u\Phi(u) > 0$ for $u \neq 0$,
(c) $\Phi \in C^1(\mathbb{R} \setminus \{0\})$ and $\Phi'(u) = r|u|^{r-1} > 0$ for $u \neq 0$, and hence Φ^{-1} exists and is strictly increasing on $\mathbb{R}$.

A function $x = x(t)$ is said to be a solution of Eq. (1.1) on $\mathbb{R}_+$ if $x, p\Phi(x') \in AC_{\text{loc}}(\mathbb{R}_+)$, the set of locally absolutely continuous functions on $\mathbb{R}_+$, such that Eq. (1.1) is satisfied a.e. on $\mathbb{R}_+$. A solution is said to be oscillatory on $\mathbb{R}_+$ if it has arbitrarily large zeros, and nonoscillatory otherwise. It is well known that all solutions of Eq. (1.1) can be extended to $\mathbb{R}_+$, and all solutions are oscillatory if and only if one solution is oscillatory. Therefore, Eq. (1.1) can be classified as either oscillatory or nonoscillatory.

When $r = 1$ and $p(t) \equiv 1$, Eq. (1.1) reduces to the linear equation

$$x'' + q(t)x = 0, \tag{1.2}$$

where $q \in L_{\text{loc}}(\mathbb{R}_+)$ and $q \geqslant 0$ a.e. on $\mathbb{R}$. In [2], Huang established the following nonoscillation and oscillation criteria for Eq. (1.2):

Theorem 1.1. *Let* $\alpha_0 = 3 - 2\sqrt{2}$.

(i) *If there exists* $t_0 > 0$ *such that for all* $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt \leqslant \frac{\alpha_0}{2^{n+1} t_0},$$

then Eq. (1.2) *is nonoscillatory.*

(ii) *If there exist* $t_0 > 0$ *and* $\alpha > \alpha_0$ *such that for all* $n \in \mathbb{N}_0$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt \geqslant \frac{\alpha}{2^n t_0},$$

then Eq. (1.2) *is oscillatory.*

By replacing the sequence $\{2^n\}$ in Theorem 1.1 by $\{\lambda^n\}$ with $\lambda > 1$, Wong [4] generalized Huang's criteria as follows:

Theorem 1.2. *Let $\lambda > 1$ and $\alpha_0 = (\sqrt{\lambda} - 1)^2$.*

(i) *If there exists $t_0 > 0$ such that for all $n \in \mathbb{N}_0$,*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)\,dt \leqslant \frac{\alpha_0}{(\lambda - 1)\lambda^{n+1} t_0},$$

then Eq. (1.2) *is nonoscillatory.*

(ii) *If there exist $t_0 > 0$ and $\alpha > \alpha_0$ such that for all $n \in \mathbb{N}_0$,*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)\,dt \geqslant \frac{\alpha}{(\lambda - 1)\lambda^{n} t_0},$$

then Eq. (1.2) *is oscillatory.*

In a different direction, Elbert [1] generalized Huang's criteria to get the results below:

Theorem 1.3. *Let $\{t_n\}$ be an strictly increasing sequence in $\mathbb{R}_+$ such that $t_n \to \infty$, and define $\beta_n = (t_{n+1} - t_n)/(t_1 - t_0)$ and $\theta_n = \beta_n/\beta_{n+1}$, $n \in \mathbb{N}_0$.*

(i) *Assume for each $n \in \mathbb{N}_0$ there exists $\alpha_n \in [0, 1)$ such that*

$$(t_{n+1} - t_n) \int_{t_n}^{t_{n+1}} q(t)\,dt \leqslant \alpha_n,$$

and the sequence $\{z_n\}$ defined by

$$z_0 = 1, \qquad z_{n+1} = \frac{z_n - \alpha_n}{\theta_n + z_n - \alpha_n}, \quad n \in \mathbb{N}_0,$$

satisfies $z_n \in (0, 1)$ for all $n \in \mathbb{N} := \{1, 2, \ldots\}$. Then Eq. (1.2) *is nonoscillatory.*

(ii) *Assume for each $n \in \mathbb{N}_0$ there exists $\alpha_n > 0$ such that*

$$(t_{n+1} - t_n) \int_{t_n}^{t_{n+1}} q(t)\,dt \geqslant \alpha_n,$$

and the sequence $\{v_n\}$ defined by

$$v_0 = 0, \qquad v_{n+1} = \frac{\alpha_{n+1}\theta_n}{\alpha_n}\left(\alpha_n + \frac{v_n}{1 - v_n}\right), \quad n \in \mathbb{N}_0,$$

does not satisfy that $v_n \in (0, 1)$ for all $n \in \mathbb{N}$. Then Eq. (1.2) *is oscillatory.*

Recently, people have tried to extend the above results to second order half-linear equations, especially to Eq. (1.1) with $p \equiv 1$, i.e., the equation

$$\big[\Phi(x')\big]' + q(t)\Phi(x) = 0. \tag{1.3}$$

In an effort of extending Huang's original criteria, Jiang [3] claimed the following:

Theorem 1.4.

(i) *Suppose* $0 < r \leqslant 1$. *If there exist* $t_0 > 0$ *and* $c \in (0,1)$ *such that for all* $n \in \mathbb{N}_0$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt \leqslant \frac{\alpha_0}{r c^{r-1} (2^{n+1} t_0)^r}, \tag{1.4}$$

where $\alpha_0 = 3 - 2\sqrt{2}$, *then Eq.* (1.3) *is nonoscillatory.*

(ii) *Suppose* $r \geqslant 1$. *If there exist* $t_0 > 0$, $c \in (0,1)$, *and* $\alpha > 2^r + 1 - 2^{1+r/2}$ *such that for all* $n \in \mathbb{N}_0$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt \geqslant \frac{\alpha}{r c^{r-1} (2^n t_0)^r}, \tag{1.5}$$

then Eq. (1.3) *is oscillatory.*

However, the constant c used in the proofs depends on each individual solution and on each value of t. Thus, there is no uniform c for all solutions and for all values of t. More importantly, the c involved in the proofs cannot be the same as the ones given in the assumptions of the theorem. Therefore, the results in Theorem 1.4 failed to be justified, and by my judgment, are unlikely to be true.

Yang [6] derived an extension of Elbert's criteria to Eq. (1.3). However, the results were not accurately stated in [6]. The following is the corrected version of Yang's results:

Theorem 1.5. *Let* $\{t_n\}$ *be a strictly increasing sequence in* $\mathbb{R}_+$ *such that* $t_n \to \infty$, *and define* $\beta_n = (t_{n+1} - t_n)/(t_1 - t_0)$ *and* $\theta_n = (\beta_n/\beta_{n+1})^r$, $n \in \mathbb{N}_0$.

(i) *Assume* $0 < r \leqslant 1$ *and for each* $n \in \mathbb{N}_0$ *there exists* $\alpha_n \in [0,1)$ *such that*

$$(t_{n+1} - t_n)^r \int_{t_n}^{t_{n+1}} q(t)\,dt \leqslant \alpha_n,$$

and the sequence $\{z_n\}$ *defined by*

$$z_0 = 1, \qquad z_{n+1} = \frac{z_n - \alpha_n}{\theta_n + z_n - \alpha_n}, \quad n \in \mathbb{N}_0,$$

satisfies $z_n \in (0,1)$ *for all* $n \in \mathbb{N}$. *Then Eq.* (1.3) *is nonoscillatory.*

(ii) *Assume* $r \geqslant 1$ (*missed in the original version*) *and for each* $n \in \mathbb{N}_0$ *there exists* $\alpha_n > 0$ *such that*

$$(t_{n+1} - t_n)^r \int_{t_n}^{t_{n+1}} q(t)\,dt \geqslant \alpha_n,$$

and the sequence $\{v_n\}$ *defined by*

$$v_0 = 0, \qquad v_{n+1} = \frac{\alpha_{n+1}\theta_n}{\alpha_n}\left(\alpha_n + \frac{v_n}{1 - v_n}\right), \quad n \in \mathbb{N}_0,$$

does not (*missed in the original version*) *satisfy that* $v_n \in (0,1)$ *for all* $n \in \mathbb{N}$. *Then Eq.* (1.3) *is oscillatory.*

We further comment that in [3,6], no criteria were found for Eq. (1.3) to be nonoscillatory for the case when $r > 1$, nor for Eq. (1.3) to be oscillatory for the case when $0 < r < 1$. This was due to the restriction of applications of Lemma 2 in [6]. Yang [6] also intended to show by examples that neither part (i) nor part (ii) of Theorem 1.5 holds for all $r \in (0,\infty)$. But these examples failed to work since they do not satisfy all conditions of Theorem 1.5. In particular, the condition that $z_n \in (0,1)$ for all $n \in \mathbb{N}$ is not satisfied by Examples 1 and 2.

In this paper, we first extend Wong's criteria for the linear equation (1.2) to the special half-linear equation (1.3) with $p \equiv 1$, and then by a transformation of independent variable, to the general half-linear equation (1.1). Since we do not employ Lemma 2 in [6], we establish criteria for both nonoscillation and oscillation for all values of $r \in (0,\infty)$. Therefore, our work applies to the general equation (1.1) and provides a complete extension of Wong's criteria. The approach in this paper can also to used to obtain an complete extension of Elbert's criteria for all values of $r \in (0,\infty)$ which will cover and improve the partial extension by Yang [6], but we omit the details.

This paper is organized as follows: the main results are presented in Section 2, and their proofs are given in Section 3 after some technical lemmas are derived.

2. Main results

To present our main results, we need to utilize the function defined as follows: for $\lambda > 1$, define

$$f(x,r,\alpha) = \frac{1}{\lambda}\left(\alpha + \frac{x}{(1-x^r)^{1/r}}\right) \quad \text{for } x \in [0,1),\ r \in (0,\infty),\ \alpha \in (0,\lambda). \tag{2.1}$$

By a simple computation we see that

(i) for fixed r and α, f is strictly increasing and concave up in x;

(ii) for fixed x and α, f is strictly decreasing in r, and

$$\lim_{r\to 0} f(x,r,\alpha) = \begin{cases} \alpha/\lambda, & x = 0, \\ \infty, & x \in (0,1), \end{cases} \quad \text{and} \quad \lim_{r\to\infty} f(x,r,\alpha) = (x+\alpha)/\lambda, \quad x \in [0,1).$$

It is clear that f has a fixed point $x^* \in (0,1)$ if and only if the curve $y = f(x,r,\alpha)$ intersects the line $y = x$ at $x^* \in (0,1)$. For fixed r, the largest value of α that makes this to happen is the one which makes the curve $y = f(x,r,\alpha)$ tangent to the line $y = x$ at some point $x^* \in (0,1)$. Such an α exists for each $r \in (0,\infty)$ since f is strictly increasing and concave up in x. In the sequel, we denote by $\alpha^* = \alpha^*(r)$ such value of α for a given r. Then we have the following:

(iii) Let $r \in (0,\infty)$ be fixed. Then $f(x^*,r,\alpha^*) = x^*$ for a unique $x^* \in (0,1)$ and $0 < \alpha^*/\lambda < x^* < 1$; and for $\alpha > \alpha^*$, $f(x,r,\alpha) > x$ for all $x \in [0,1)$, and hence f has no fixed point in $[0,1)$; see Fig. 1.

(iv) $\alpha^*(r)$ is continuous and strictly increasing on $(0,\infty)$, and satisfies

$$\lim_{r\to 0}\alpha^*(r) = 0 \quad \text{and} \quad \lim_{r\to\infty}\alpha^*(r) = \lambda - 1;$$

this is implied in Fig. 2.

Although $\alpha^*(r)$ cannot be computed analytically in general, it can be computed for the linear case. In fact, $\alpha^*(1) = (\sqrt{\lambda}-1)^2$ as shown in [4]. For the case when $r \neq 1$, $\alpha^*(r)$ can be evaluated numerically.

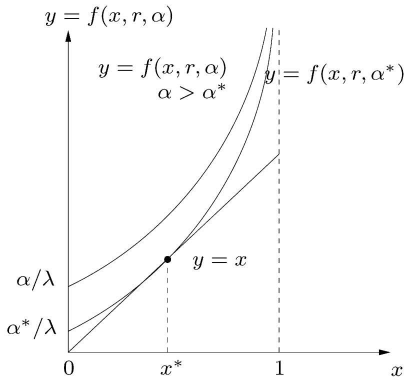

Fig. 1.

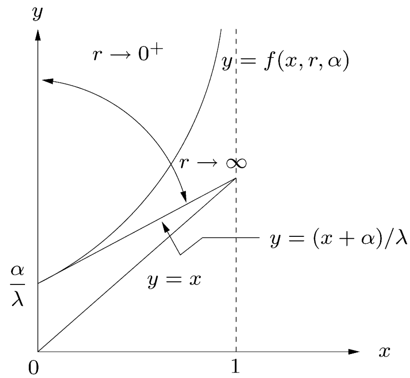

Fig. 2.

Now we state our main results.

Theorem 2.1. *Let $\lambda > 1$ and $\alpha^* = \alpha^*(r)$. Assume there exists $t_0 \in (0, \infty)$ such that for each $n \in \mathbb{N}_0$,*

$$(t_{n+1} - t_n)\left(\int_{t_n}^{t_{n+1}} q(t)\,dt\right)^{1/r} \leqslant \frac{\alpha^*}{\lambda}, \tag{2.2}$$

where $t_n = \lambda^n t_0$. Then Eq. (1.3) *is nonoscillatory.*

Theorem 2.2. *Let $\lambda > 1$ and $\alpha^* = \alpha^*(r)$. Assume there exist $t_0 \in (0, \infty)$ and $\alpha > \alpha^*$ such that for each $n \in \mathbb{N}_0$,*

$$(t_{n+1} - t_n)\left(\int_{t_n}^{t_{n+1}} q(t)\,dt\right)^{1/r} \geqslant \alpha, \tag{2.3}$$

where $t_n = \lambda^n t_0$. Then Eq. (1.3) *is oscillatory.*

The above results can be extended to the general equation (1.1) via a change of the independent variable. Let

$$\tau = g(t) := \int_a^t p^{-1/r}(s)\,ds. \tag{2.4}$$

Then Eq. (1.1) becomes

$$\frac{d}{d\tau}\left(\Phi\left(\frac{dy(\tau)}{d\tau}\right)\right) + \bar{q}(\tau)\Phi\big(y(\tau)\big) = 0, \tag{2.5}$$

where $\bar{q}(\tau) = p^{1/r}(t(\tau))q(t(\tau))$. For $t_0 \in (0, \infty)$, let $\tau_0 = \int_0^{t_0} p^{-1/r}(t)\,dt$ and $\tau_n = \lambda^n \tau_0$. Then we can apply Theorems 2.1 and 2.2 to Eq. (2.5) with q, t_0, t_n replaced by $\bar{q}$, τ_0, τ_n, respectively, to obtain the following results for Eq. (1.1).

Theorem 2.3. *Let $\lambda > 1$ and $\alpha^* = \alpha^*(r)$. Assume there exists $t_0 \in (0, \infty)$ such that for each $n \in \mathbb{N}_0$,*

$$\int_{t_n}^{t_{n+1}} p^{-1/r}(t)\,dt \left(\int_{t_n}^{t_{n+1}} q(t)\,dt \right)^{1/r} \leqslant \frac{\alpha^*}{\lambda}, \tag{2.6}$$

where $t_n = g^{-1}(\lambda^n \int_0^{t_0} p^{-1/r}(t)\,dt)$ with $g(t)$ given in (2.4)*. Then Eq.* (1.1) *is nonoscillatory.*

Theorem 2.4. *Let $\lambda > 1$ and $\alpha^* = \alpha^*(r)$. Assume there exist $t_0 \in (0, \infty)$ and $\alpha > \alpha^*$ such that for each $n \in \mathbb{N}_0$,*

$$\int_{t_n}^{t_{n+1}} p^{-1/r}(t)\,dt \left(\int_{t_n}^{t_{n+1}} q(t)\,dt \right)^{1/r} \geqslant \alpha, \tag{2.7}$$

where $t_n = g^{-1}(\lambda^n \int_0^{t_0} p^{-1/r}(s)\,ds)$ with $g(t)$ given in (2.4)*. Then Eq.* (1.1) *is oscillatory.*

3. Proofs

To prove our main results we need to introduce several lemmas. The first one was given by Yang [6] which provides an extension of Wintner's lemma in [5] for the linear equation (1.2) to the half-linear equation (1.3).

Lemma 3.1. *Let $0 \leqslant a < b$ and $x(t)$ a nontrivial solution of Eq.* (1.3) *satisfying either $x(a) = x'(b) = 0$ or $x'(a) = x(b) = 0$. Then*

$$(b-a)\left(\int_a^b q(t)\,dt \right)^{1/r} > 1.$$

To prove Theorem 2.1 we need the following two lemmas on the behavior of certain solutions of Eq. (1.3) for the case when (2.2) holds.

Lemma 3.2. *Let $\lambda > 1$ and $t_k = \lambda^k t_0, k \in \mathbb{N}_0$, and assume* (2.2) *holds. Suppose there exists $n \in \mathbb{N}_0$ such that $x(t)$ is a solution of Eq.* (1.3) *satisfying $x(t_0) = 0$, $x(t) > 0$ on $[t_0, t_{n+1}]$, and $x'(t) > 0$ on $[t_0, t_n]$. Then*

$$\Phi\big(x'(t_{n+1})\big) \geqslant \Phi\big(x'(t_n)\big) - \Phi\left(\sum_{i=0}^{n} \frac{\alpha^*}{\lambda^{n+1-i}} x'(t_i) \right). \tag{3.1}$$

Proof. From Eq. (1.3), $\Phi(x'(t))$ is decreasing on $[t_0, t_{n+1}]$, so is $x'(t)$. Then for $t \in [t_n, t_{n+1}]$,

$$\begin{aligned} x(t) &= x(t) - x(t_n) + \sum_{i=0}^{n-1}\big(x(t_{i+1}) - x(t_i)\big) = \int_{t_n}^{t} x'(s)\,ds + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} x'(s)\,ds \\ &\leqslant x'(t_n)(t - t_n) + \sum_{i=0}^{n-1} x'(t_i)(t_{i+1} - t_i) \leqslant \sum_{i=0}^{n} x'(t_i)(t_{i+1} - t_i). \end{aligned} \tag{3.2}$$

From (1.3), (3.2), and the definition of t_n,

$$\begin{aligned}
\Phi\big(x'(t_n)\big) - \Phi\big(x'(t_{n+1})\big) &= \int_{t_n}^{t_{n+1}} q(t)\Phi\big(x(t)\big)\,dt \\
&\leqslant \int_{t_n}^{t_{n+1}} q(t)\Phi\left(\sum_{i=0}^{n} x'(t_i)(t_{i+1}-t_i)\right)dt \\
&= \Phi\left(\sum_{i=0}^{n} x'(t_i)(t_{i+1}-t_i)\right)\int_{t_n}^{t_{n+1}} q(t)\,dt \\
&= \Phi\left(\sum_{i=0}^{n} x'(t_i)(t_{i+1}-t_i)\left(\int_{t_n}^{t_{n+1}} q(t)\,dt\right)^{1/r}\right) \\
&= \Phi\left(\sum_{i=0}^{n} x'(t_i)\frac{t_{i+1}-t_i}{t_{n+1}-t_n}(t_{n+1}-t_n)\left(\int_{t_n}^{t_{n+1}} q(t)\,dt\right)^{1/r}\right) \\
&= \Phi\left(\sum_{i=0}^{n} \frac{1}{\lambda^{n-i}} x'(t_i)(t_{n+1}-t_n)\left(\int_{t_n}^{t_{n+1}} q(t)\,dt\right)^{1/r}\right).
\end{aligned}$$

Using (2.2) we find that

$$\Phi\big(x'(t_n)\big) - \Phi\big(x'(t_{n+1})\big) \leqslant \Phi\left(\sum_{i=0}^{n} \frac{\alpha^*}{\lambda^{n+1-i}} x'(t_i)\right).$$

Therefore, (3.1) holds. □

To state Lemma 3.3, we introduce a sequence $\{l_k\}_{k=0}^{\infty}$ based on the function f defined in (2.1). For $\lambda > 1$ and $\alpha^* = \alpha^*(r)$ let

$$l_0 = \alpha^*/\lambda, \quad \text{and} \quad l_{k+1} = f\big(l_k, r, \alpha^*\big), \quad k \in \mathbb{N}_0. \tag{3.3}$$

Since f is strictly increasing in x and $l_0 < x^* < 1$ for the unique fixed point x^* of $f(\cdot, r, \alpha^*)$, it is easy to see that $\{l_k\}$ is well defined, strictly increasing, and $0 < l_k < x^* < 1$ for $n \in \mathbb{N}_0$. Therefore, $\lim_{k\to\infty} l_k = l^*$ exists and hence is a fixed point of f. By the uniqueness of the fixed point of $f(\cdot, r, \alpha^*)$ in (0, 1), we see that $l^* = x^*$.

Lemma 3.3. *Let the assumptions of Lemma* 3.2 *hold with a fixed $n \in \mathbb{N}_0$ and $\{l_k\}$ be defined as in* (3.3). *Then for $k = 0, 1, \ldots, n+1$,*

$$l_k x'(t_k) \geqslant \sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+1-i}} x'(t_i). \tag{3.4}$$

Proof. We prove it by induction. Note that $l_0 = \alpha^*/\lambda$ implies that (3.4) holds for $k = 0$. Assume (3.4) holds for some $k \in \{0, 1, \ldots, n\}$. By Lemma 3.2

$$\begin{aligned}\Phi\big(x'(t_{k+1})\big) &\geqslant \Phi\big(x'(t_k)\big) - \Phi\left(\sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+1-i}} x'(t_i)\right)\\ &= \frac{1}{l_k^r}\Phi\big(l_k x'(t_k)\big) - \Phi\left(\sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+1-i}} x'(t_i)\right).\end{aligned}$$

By the inductive assumption

$$\Phi\big(x'(t_{k+1})\big) \geqslant \left(\frac{1}{l_k^r} - 1\right)\Phi\left(\sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+1-i}} x'(t_i)\right).$$

Taking Φ^{-1} on both sides we obtain

$$x'(t_{k+1}) \geqslant \frac{(1-l_k^r)^{1/r}}{l_k} \sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+1-i}} x'(t_i).$$

Then from (3.3)

$$\begin{aligned}l_{k+1}x'(t_{k+1}) &= \frac{1}{\lambda}\left(\alpha^* + \frac{l_k}{(1-l_k^r)^{1/r}}\right)x'(t_{k+1})\\ &\geqslant \frac{\alpha^*}{\lambda}x'(t_{k+1}) + \sum_{i=0}^{k} \frac{\alpha^*}{\lambda^{k+2-i}} x'(t_i) = \sum_{i=0}^{k+1} \frac{\alpha^*}{\lambda^{k+2-i}} x'(t_i).\end{aligned}$$

Therefore, (3.4) holds for $k+1$. $\square$

Proof of Theorem 2.1. Let $x(t)$ be a solution of Eq. (1.3) satisfying $x(t_0) = 0$ and $x'(t_0) > 0$. We prove by induction that for any $n \in \mathbb{N}_0$,

$$x'(t) > 0 \quad \text{on } [t_0, t_n] \quad \text{and} \quad x(t) > 0 \quad \text{on } [t_0, t_{n+1}]. \tag{3.5}$$

First we claim that $x'(t) > 0$ on $[t_0, t_1]$. For otherwise, there exists $s_0 \in (t_0, t_1]$ such that $x'(s_0) = 0$. By Lemma 3.1 and (2.2) with $n = 0$ we have

$$1 < (s_0 - t_0)\left(\int_{t_0}^{s_0} q(t)\,dt\right)^{1/r} \leqslant (t_1 - t_0)\left(\int_{t_0}^{t_1} q(t)\,dt\right)^{1/r} \leqslant \alpha^*/\lambda$$

contradicting the fact that $\alpha^* < \lambda$. Therefore, $x(t) > x(t_0) > 0$ for $t \in (t_0, t_1]$. This verifies (3.5) for $n = 0$.

Suppose that (3.5) holds for some $n \in \mathbb{N}_0$. From Lemmas 3.2 and 3.3

$$\begin{aligned}\Phi\big(x'(t_{n+1})\big) &\geqslant \Phi\big(x'(t_n)\big) - \Phi\big(l_n\big(x'(t_n)\big)\big) = \Phi\big(x'(t_n)\big) - l_n^r\Phi\big(x'(t_n)\big)\\ &= \big(1 - l_n^r\big)\Phi\big(x'(t_n)\big) > 0\end{aligned}$$

which implies that $x'(t_{n+1}) > 0$. Since $x(t) > 0$ on $[t_0, t_{n+1}]$, from (1.3) we see that $\Phi(x'(t))$ is decreasing on $[t_0, t_{n+1}]$, so is $x'(t)$. Thus, $x'(t) \geqslant x'(t_{n+1}) > 0$ on $[t_0, t_{n+1}]$.

Finally, we show that $x(t) > 0$ on $[t_0, t_{n+2}]$. Assume the contrary, then there exists $\tau \in (t_{n+1}, t_{n+2}]$ such that $x(\tau) = 0$. Since $x'(t_{n+1}) > 0$, there exists $s \in (t_{n+1}, \tau)$ such that $x'(s) = 0$. Using Lemma 3.1 on the interval $[s, \tau]$ and applying (2.2) we have

$$1 < (\tau - s)\left(\int_s^\tau q(t)\,dt\right)^{1/r} \leqslant (t_{n+2} - t_{n+1})\left(\int_{t_{n+1}}^{t_{n+2}} q(t)\,dt\right)^{1/r} \leqslant \alpha^*/\lambda$$

contradicting the fact that $\alpha^* < \lambda$. Hence $x(t) > 0$ on $[t_0, t_{n+2}]$. By the arbitrariness of $n \in \mathbb{N}_0$, we see that $x(t)$ is nonoscillatory. Therefore, Eq. (1.3) is nonoscillatory. $\square$

To prove Theorem 2.2 we need the following lemma on the behavior of certain solutions of Eq. (1.3) for the case when (2.3) holds.

Lemma 3.4. *Let $\lambda > 1$ and $t_k = \lambda^k t_0$, $k \in \mathbb{N}_0$, and assume* (2.3) *holds for some $\alpha > \alpha^*$. Suppose there exists $n \in \mathbb{N}$ such that $x(t)$ is a solution of Eq.* (1.3) *satisfying $x(t) > 0$ and $x'(t) > 0$ on $[t_0, t_{n+1}]$. Then*

$$\Phi\big(x'(t_{n+1})\big) \leqslant \Phi\big(x'(t_n)\big) - \Phi\left(\sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i)\right). \tag{3.6}$$

The proof of Lemma 3.4 is similar to that of Lemma 3.2 and hence is omitted.

Proof of Theorem 2.2. We prove it by contradiction. Let $x(t)$ be a nonoscillatory solution of Eq. (1.3), say $x(t) > 0$ on $[t_0, \infty)$. By (1.3), $\Phi(x'(t))$ is decreasing on $[t_0, \infty)$, so is $x'(t)$. This implies that $x'(t) > 0$ on $[t_0, \infty)$.

Without loss of generality assume $\alpha \in (\alpha^*, \lambda)$. We introduce a sequence $\{h_n\}_{n=1}^{\infty}$ based on the function f defined in (2.1) as follows:

$$h_1 = \alpha/\lambda, \quad \text{and} \quad h_{n+1} = f(h_n, r, \alpha), \quad n \in \mathbb{N}.$$

We show that $h_n \in (0, 1)$ for all $n \in \mathbb{N}$ and hence the sequence $\{h_n\}$ is well defined. More specifically, we prove that

$$0 < h_n x'(t_n) \leqslant \sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i) < x'(t_n), \quad n \in \mathbb{N}, \tag{3.7}$$

which implies that $h_n \in (0, 1)$ for $n \in \mathbb{N}$. Obviously, (3.7) holds with $n = 1$. Assume (3.7) holds for some $n \in \mathbb{N}$. By Lemma 3.4, (3.6) is satisfied. From (3.6) and the inductive assumption

$$\begin{aligned}
\Phi\big(x'(t_{n+1})\big) &\leqslant \Phi\big(x'(t_n)\big) - \Phi\left(\sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i)\right) \\
&= \frac{1}{h_n^r} \Phi\big(h_n x'(t_{n+1})\big) - \Phi\left(\sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i)\right) \\
&\leqslant \frac{1 - h_n^r}{h_n^r} \Phi\left(\sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i)\right).
\end{aligned}$$

This implies that

$$x'(t_{n+1}) \leqslant \frac{(1-h_n^r)^{1/r}}{h_n} \sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i).$$

Since $h_n \in (0,1)$, h_{n+1} is well defined and $h_{n+1} > 0$. Then

$$\begin{aligned} h_{n+1}x'(t_{n+1}) &= \frac{1}{\lambda}\left(\alpha + \frac{h_n}{(1-h_n^r)^{1/r}}\right)x'(t_{n+1}) \\ &\leqslant \frac{\alpha}{\lambda}x'(t_{n+1}) + \sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+2-i}} x'(t_i) = \sum_{i=1}^{n+1} \frac{\alpha}{\lambda^{n+2-i}} x'(t_i). \end{aligned}$$

Therefore, the left part of (3.7) holds for $n+1$. Note that $x'(t_{n+1}) > 0$. From (3.6) we have

$$\Phi\left(\sum_{i=1}^{n} \frac{\alpha}{\lambda^{n+1-i}} x'(t_i)\right) < \Phi\big(x'(t_i)\big).$$

This shows that the right part of (3.7) also holds.

Note that f is increasing in x and $h_n \in (0,1)$. The sequence $\{h_n\}$ is increasing and bounded. Thus, $\lim_{n\to\infty} h_n = h^*$ exists and h^* is a fixed point of the function f. This implies that $h^* \in (0,1)$ which contradicts the fact that f has no fixed point in $(0,1)$ for $\alpha > \alpha^*$. □

Proof of Theorems 2.3 and 2.4. Since $\int_0^\infty p^{-1/r}(t)\,dt = \infty$, we have $\lim_{t\to\infty} \tau(t) = \infty$. Clearly, Eq. (1.1) is oscillatory if and only if Eq. (2.5) is oscillatory. Note that $\tau_n = \lambda^n \tau_0$ and $\tau_n = g(t_n) = \int_0^{t_n} p^{-1/r}(s)\,ds$. Hence

$$\tau_{n+1} - \tau_n = \int_{\tau_n}^{\tau_{n+1}} \bar{q}(\tau)\,d\tau = \int_{g(t_n)}^{g(t_{n+1})} \bar{q}(\tau)\,d\tau = \int_{t_n}^{t_{n+1}} q(t)\,dt.$$

Now, conditions (2.6) and (2.7) imply conditions (2.2) and (2.3), respectively. Then the conclusions follow from Theorems 2.1 and 2.2. □

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